Content

Tutorial 9
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BACK

Questions of today

1. There are some generalizations of Schwarz lemma:
   a. Let f be holomorphic on $\mathbb{D}$ and suppose that $|f(z)| \le M$ for all $z \in \mathbb{D}$. Suppose $\alpha_1, \alpha_2, \dots \alpha_n \in \mathbb{D}$ with $f(z_k) = 0$. Show that
   $$|f(z)| \le M \prod_{k=1}^{n} \frac{|z-\alpha_k|}{|1-\overline{\alpha_k} z|}$$
   b. (Schwarz-Pick) Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic. For $a, b \in \mathbb{D}$, show that
   $$\left| \frac{f(a)-f(b)}{1-\overline{f(a)}f(b)} \right| \le \left| \frac{a-b}{1-\bar{a}b} \right|.$$
2. There are some statements making use of the fact that the Caylay transform
   $$z \mapsto \frac{z-i}{z+i}$$
   from the upper half plane $\mathbb{H}$ to the unit disc $\mathbb{D}$.
   a. Any entire map f with $\mathrm{Re}(f)$ bounded below is constant.
   b. Let $\mathcal{F}$ be a family of function on a region Ω such that the real parts of $\mathcal{F}$ are bounded below, then $\mathcal{F}$ is normal.
   c. (Borel–Carathéodory) Let f be a holomorphic function defined on the closed unit disc $D_R = \{z : |z| \le R\}$. Show that, for $r < R$,
   $$\sup_{|z|\le r} |f(z)| \le \frac{2r}{R-r} \sup_{|z|\le R} \mathrm{Re} f + \frac{R+r}{R-r} f(0).$$
3. Suppose $\{f_n\}$ is a sequence of holomorphic functions on Ω, and $f_n \to f$ uniformly on compact subset. Show that f is holomorphic and $f_n^{(k)} \to f^{(k)}$ for any positive integral k.
4. (Hurwitz) Let Ω be a region (so it is connected), and $\{f_n\}$ a sequence of nonvanishing holomorphic functions on Ω. Suppose $f_n \to f$ uniformly on compact subsets, show that either $f \equiv 0$ or f is nowhere vanishing.
5. (Vitali) Let $\{f_n\}$ be a locally bounded sequence of holomorphic functions on Ω, and f is a holomorphic function on Ω. Suppose the set $A = \{z \in \Omega : f_n(z) \to f(z)\}$ has a limit point in Ω, show that $f_n \to f$ uniformly on compact subsets.

Hints & solutions of today

1. a. Replace f with f/M, we may assume $M = 1$. Let
   $$\phi_\alpha(z) = \frac{z-\alpha}{1-\bar{\alpha}z}.$$
   Consider the function f/ϕ_{α_n}, we may by induction assume $n = 1$. Finally, consider the function $f \circ \phi_{\alpha_1}^{-1}$ and apply the Schwarz lemma.
   b. This is homework 4.
2. a. By composing with the (together with a tranlation) Caylay transform, the assumption becomes $|f|$ is bounded. Then apply Liouville theorem.
   b. Composing with the (together with a tranlation)Caylay transform, we may assume the family is uniformly bounded.
   c. As above, composing with a series of conformal map, we translate f to a function F which maps $\mathbb{D}$ to $\mathbb{D}$, then apply Schwarz Lemma to F.
3. Holomorphicity can be seen by Goursat theorem. Uniform convergence of derivative can be seen by Cauchy's estimate.
4. Suppose f is 0 at some point a but f is not identically zero. Find a small disc D aroud a so that a is the only zero of f in the D. Then apply Rouche theorem to f and f_n for large n to get a contradiction.
5. Suppose on the contrary that there exists a compact subset K that f_n does not converge uniformly on K. That is, by replacing f_n with a subsequence, we may assume there exists $\epsilon > 0$ in K such that $\sup ||f(a_n) - f_n(a_n)||_K \ge \epsilon$. On the other hand, by passing to a subsequence, we may use Montel theorem to assume f_n converges uniformlly on compact subsets to a holomorphic function g. Now $g = f$ on A so we have $f \equiv g$ by the identity theorem, we thus get a contradiction.
